Modes of convergence : Theorems, Examples and Counterexamples

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These notes are mostly from [1]. All our functions are going to be measurable, so we will not explicitly specify that. Let $(X, \mathcal{B}, \mu)$ be a measured space and f_n be real or complex valued measurable functions on X and f be another such function. Often, when the measure is clear (we will anyway not need more than one measure), we write $\int_X f$ for $\int_X f d\mu$. We say

- Pointwise convergence: $f_n \to f$ pointwise almost everywhere (abbreviated a.e.) if for almost all $x \in X, f_n(x)$ converges to $f(x)$.
- Convergence in L^p: f_n converge to f in L^p if the L^p norm $\|f_n - f\|_p = (\int_X |f_n - f|^p d\mu)^{1/p}$ converges to zero. Here $p \in [1, \infty)$.
- Convergence in L^∞: This is similar to the previous convergence. If g is a function on X, it is essentially bounded by $M > 0$ if $\{x : |g(x)| > M\}$ has measure zero. The L^∞ norm is defined as $\|g\|_\infty = \inf\{M : g \text{ is essentially bounded by } M\}$.
- Almost uniform convergence: This is a generalization of uniform convergence. f_n converge almost uniformly to f if for every $\epsilon > 0$ there is a measurable set E of measure at most ϵ such that f_n converge to f uniformly on $X \setminus E$.
- Convergence in measure: f_n converge in measure to f if for every $\epsilon > 0$, the measures $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}$ converge to zero as $n \to \infty$.

Theorem 1. *(Monotone Convergence Theorem) Let f_n be a non decreasing sequence of non negative measurable functions. Define $f(x) = \lim f_n(a) = \sup f_n(x) \in [0, \infty]$. Then f is measurable and*

$$\int_X f d\mu = \lim_{n\to\infty} f_n d\mu.$$

Theorem 2. *(Fatou's lemma) Let f_n be a sequence of non negative measurable functions, then*

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu.$$

In particular, if $f_n \to f$ pointwise a.e., then $\int_X f d\mu \leq \liminf \int_X f_n d\mu$.

Theorem 3. *(Dominated Convergence Theorem) Let f_n be a sequence of real or complex valued functions on X convergine pointwise a.e. to a function f. Then f is measurable. Further suppose $g \in L^1(\mu)$ is a non negative function dominating all f_n, i.e., $|f_n(x)| \leq g(x)$ a.e., then f is in $L^1(\mu)$ and $\int_X f d\mu = \lim \int_X f_n d\mu$. Additionally, f_n converge to f in L^1.*

Some counter-examples:

- Pointwise convergence does not imply L^1 convergence : (Escape to vertical infinity) Consider the sequence $n\chi_{[0,1/n]}$ on $[0, 1]$ converging pointwise to 0 almost everywhere, but all the integrals are 1, not converging to zero.
  We also have width infinity, consider $\chi_{[0,n]}/n$ converging pointwise to 0 on $\mathbb{R}$, but not in L^1 and escape to infinity with $1\chi_{[n,n+1]}$ converging to zero pointwise but not in L^1.

- L^1 convergnce does not imply pointwise a.e. convergence : Consider the typewriter sequence $\chi_{[0,1/n]}, \chi_{[1/n,2/n]}, \ldots, \chi_{[1-1/n,1]}$ which converge to zero in L^1 norm, but not pointwise because at every point the sequence is 1 and 0 infinitely often.

Some implications:

- Clearly uniform convergence implies almost uniform convergence, and pointwise convergence.
- If f_n converge almost uniformly to f, then obtain the set E_m of measure at most $1/2^m$ outside which f_n converge uniformly, hence pointwise, to f. If $x \notin \cap_{N\geq 1} \cup_{n\geq N} E_n$, then $f_n(x) \to f(x)$ and it is clear that this exception set has measure zero, therefore f_n converge a.e. to f.
- If f_n converge uniformly to f, then clearly f_n converge in L^∞ to f. Conversely, suppose they converge in L^∞. Given $1/m$, there is an integer N_m, such that for $n \geq N_m, |f_n - f|$ is essentially bounded by $2/m$, i.e., for each $n \geq N_m$, there is a measure zero set $E_{n,m}$ such that $|f_n(x) - f(x)| < 1/m$ for $x \notin E_{n,m}$. Take $E = \cup_{m\geq 1} \cup_{n\geq N_m} E_{n,m}$, then $\mu(E) = 0$ and it is clear that f_n converge uniformly to f outside E. Therefore, f_n converge almost uniformly and uniformly outside a null set to f.
- If f_n converge to f almost uniformly, then it converges in measure for fix $\epsilon > 0$. Given any $1/m$, find a set E of measure at most $1/m$ such that f_n converge uniformly outside E. Then for some large $N, \{x : |f_n(x) - f(x)| > \epsilon\} \subseteq E \,\forall\, n \geq N$, hence $\limsup \mu\{x : |f_n(x) - f(x)| > \epsilon\} \leq 1/m$.

Theorem 4. *(Uniqueness) Suppose a sequence of measureable real/complex valued functions f_n on X converge to f along one of the modes above and to g along another, then $f = g$ a.e.*

Proof. The proof can be reduced to specific modes of convergence using the implications above. For a proof, see [1]. □

Theorem 5. *(Chebyshev-Markov inequality) Let (X, μ) be a measured space and f a measurable function. Then*

$$\mu(\{x : |f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \|f\|_1$$

for every $\epsilon > 0$.

Proof. This is a really simple inequality. Let $A = \{x : |f(x)| \geq \epsilon\}$, then A is measurable and

$$\mu(A) = \int_X \chi_A d\mu \leq \int_X \frac{1}{\epsilon}|f| d\mu = \frac{1}{\epsilon} \|f\|_1 .$$ □

Corollary 1. *If f_n converge in L^1 to f, then they converge in measure as well.*

Theorem 6. *(Egorov's theorem) Let (X, μ) be a finite measure space and f_n converge pointwise a.e. to f, then they converge almost uniformly to f.*

Proof. In order to show uniform convergence, we need to show that $|f_n - f|$ can be made uniformly small. Given N, m set

$$E_{N,m} = \{x | \exists n \geq N : |f_n(x) - f(x)| > 1/m\}.$$

Observe that for a fixed m, $E_{1,m} \supseteq E_{2,m} \supseteq \ldots$. Moreover, by a.e. pointwise convergence, the set $\cap_n E_{n,m}$ has measure zero. Since X is a finite measure space, $\mu(\cap_n E_{n,m}) = \lim_n \mu(E_{n,m}) = \inf_n E_{n,m}$.

Given $\epsilon > 0$, for every m, choose N_m such that for $n \geq N_m, \mu(E_{n,m}) \leq \epsilon/2^m$, and take $E = \cup_{m\geq 1} E_{N_m,m}$. Then, $\mu(E) \leq \epsilon$ and if $x \notin E$, then it is not in every $E_{N_m,m}$, hence for $n \geq N_m, |f_n(x) - f(x)| \leq 1/m$ and this N_m doesn't depend on x, therefore f_n converges uniformly to f outside A. □

Remark. Note that the purpose of finite measure was to obtain a downward monotone convergence. If there were other ways to obtain N_m, then the proof above holds.

Here is another implication under the assumption of finite measure.

Theorem 7. *(Comparision of L^p spaces) Suppose (X, μ) is a finite measure space. Then, for any $1 \leq p_2 \leq p_1 \leq \infty$, there is a continuous inclusion $L^{p_1}(\mu) \hookrightarrow L^{p_2}(\mu)$ with $\|f\|_{p_2} \leq \mu(X)^{1/p_2 - 1/p_1} \|f\|_{p_1}$.*

Proof. We borrow the proof from [2]. First, if $p_2 = \infty$, then there is nothing to prove, so we may take $p_2 < \infty$. If $p_1 = \infty$, then suppose $f \in L^\infty(\mu)$ and $\|f\|_\infty = M < \infty$. Then, given any $\epsilon > 0$, we have

$$\int_X |f|^{p_2} \leq \int_X (M + \epsilon) = \mu(X)(\|f\|_\infty + \epsilon) < \infty.$$

Hence, $f \in L^{p_2}(\mu)$ and $\|f\|_{p_2} \leq \mu(X)^{1/p_2}(\|f\|_\infty + \epsilon)$. Since $\epsilon > 0$ was arbitrary, we see that the inequality in the statement holds. Now we can take $p_1 < \infty$.

Given $p \in [1, \infty]$, let q be its dual exponent, i.e., a solution to $1/p + 1/q = 1$. Then, by Holder's inequality for any $f \in L^{p_1}(\mu)$,

$$\int_X |f|^{p_2} d\mu \leq \|f^{p_2}\|_p \|1\|_q = \mu(X)^{1/q} \|f^{p_2}\|_p.$$

Take $p = p_1/p_2 \geq 1$, then $q = 1 - p_2/p_1$ and the inequality above becomes

$$\|f\|_{p_2}^{p_2} \leq \|f\|_{p_1}^{p_2} \mu(X)^{1-p_2/p_1}$$

which finishes the proof. □

Corollary 2. *With the setup as above, if f_n converge in L^{p_1} to f, then they converge in L^{p_2} as well.*

Theorem 8. *(Lusin's theorem) Let $f\colon \mathbb{R}^d \to \mathbb{C}$ be absolutely integrable and let $\epsilon > 0$. Then there is a measurable $E, \lambda(E) \leq \epsilon$ such that $f|_{\mathbb{R}^d \setminus E}$ is continuous (here λ is the Lebesgue measure on $\mathbb{R}^d$).*

Proof. Using approximations theorems, one can find compactly supported continuous functions f_n such that $\|f - f_n\| \leq \epsilon/4^n$. By Chebyshev-Markov inequality,

$$\lambda(\{x : |f_n(x) - f(x)| > 1/2^{n+1}\}) \leq \epsilon/2^{n+1}$$

so, $|f(x) - f_n(x)| \leq 1/2^{n-1}$ except on a set E_n of measure $\leq \epsilon/2^{n+1}$.

Let $E = \cup E_n$, then $\lambda(E) \leq \epsilon/2$. Again, outside E, we have uniform convergence. The uniform limit of continuous functions is continuous, therefore the restriction of f is continuous outside E. □

Remark. This theorem says that the restriction is continuous, not f being continuous at points outside E. So, when $f = 1_{\mathbb{Q}}$, then $E = \mathbb{Q}$ and the restriction of f is continuous on $\mathbb{R} \setminus E$.

Lemma 1. *(Urysohn subsequence principle) A sequence $\{x_n\}$ in $\mathbb{R}$ converges to $x \in \mathbb{R}$ if and only if every subsequence has a further subsequence converging to x.*

Proof. Our proof applies more generally to Hausdorff spaces. If x_n converge to x, then it is clear that every subsequence converges to x. Next, suppose that x_n doesn't converge to x, then there is some open set U around x such that for every N, there is an $n_N \geq N$ such that $x_n \notin U$. We can obtain an infinite sequence x_{n_k} not lying in U, therefore no subsequence of this subsequence converges to x for it cannot enter U. □

Lemma 2. *(First extension of DCT) On a measured space (X, μ) suppose f_n, g_n are measureable and converging pointwise a.e. to f, g respectively and that $|f_n| \leq g_n, n \geq 1$. Further suppose $g_n \in L^1 \forall\, n \geq 1$ and $g \in L^1$ with $\int g_n \to \int g$. Then f_n converges to f in L^1.*

Proof. Because $|f_n| \leq g_n$, we have $|f| \leq g$ a.e., therefore $f \in L^1$. We have $|f_n - f| \leq g_n + g$, so define the nonnegative function $h_n = g_n + g - |f_n - f|$. Applying Fatou's lemma, we get

$$\begin{aligned}\int_X \liminf h_n &\leq \liminf \int_X h_n \\ &= \liminf \int_X g_n + \int_X g - \limsup \int_X |f_n - f|\end{aligned}$$

Now, h_n converges pointwise to $2g$ and by assumption $\int g_n \to \int g$, therefore, $\limsup \int_X |f_n - f| = 0$, hence f_n converges to f in L^1. □

Theorem 9. *Suppose on a measured space (X, μ), the sequence f_n converges in measure to a function f, then there is a subsequence f_{n_k} converging to f pointwise.*

Proof. Define $E_{n,m} = \{x : |f_n(x) - f(x)| > 1/m\}$. If we look at the sequence $\{f_n\}$, then

$$\{x : f_n(x) \not\to f(x)\} = \cup_{m\geq 1} \cap_{N\geq 1} \cup_{n\geq N} E_{n,m}.$$

By convergence in measure, $\mu(E_{n,m}) \to 0$ as $n \to \infty$ for any fixed m. For each $k \geq 1$, choose n_k such that

$$\mu(\{x : |f_n(x) - f(x)| \geq 1/2^k\}) < 1/2^k \, \forall \, n \geq n_k.$$

We may take $n_1 < n_2 < \dots$. We claim that f_{n_k} is the subsequence we are looking for.

Given an x, $f_{n_k}(x)$ doesn't converge to $f(x)$ if and only if $\epsilon_0 > 0$ (depnding on x) such that infinitely many $f_{n_k}(x)$ are outside $f(x) \pm \epsilon_0$. Choosing $2^{-N} < \epsilon_0$, we see that $x \in \cap_{N\geq 1} \cup_{k\geq N} E_k$ where $E_k = \{x : |f_{n_k}(x) - f(x)| > 1/2^k\}$. We have

$$\mu(\cap_{N\geq 1} \cup k \geq N E_k) \leq \mu(\cup_{k\geq N} E_k) \leq \sum_{k\geq N} \frac{1}{2^k} = \frac{1}{2^{N-1}}.$$

Therefore, outside a set of measure zero, f_{n_k} converges pointwise to f. □

Corollary 3. *On a measured space (X, μ), suppose functions f_n converge in measure to f and that each f_n is dominated by an absolutely integrable g, then $\int f_n \to \int f$ and $f \in L^1(\mu)$.*

Proof. By going to a subsequence if necessary, we may assume that $f_n \to f$ pointwise a.e., in which case by dominated convergence, $f \in L^1(\mu)$. Now, every subsequence of f_n converges in measure to f, hence has a subsequence converging pointwise a.e., and by dominated convergence theorem, we see that every subsequence of $\{\int_X f_n\}$ has a further subsequence converging to $\int_X f$. By Urysohn's subsequence principle, $\int_X f_n$ converges to $\int_X f$. In fact, the same argument shows that f_n converge to f in L^1. □

Lemma 3. *(Second extension of DCT; almost dominated convergence) On a measured space (X, μ), let f_n be measurable functions converging pointwise a.e. to f and let $G, g_1, g_2, \dots$ be absolutely integrable functions $X \to [0, \infty]$ such that $|f_n| \leq G + g_n$ a.e. and $\int_X g_n \to 0$. Then f_n converge to f in L^1.*

Proof. By Chebyshev-Markov inequality, it is clear that g_n converge to 0 in measure. Given a subsequence $\{f_{n_k}\}$ obtain a subsequence $\{g_{n_{k_l}}\}$ from $\{g_{n_k}\}$ that converges pointwise a.e. to 0. Then $f_{n_{k_l}}$ are bounded by $G + g_{n_{k_l}} \in L^1(\mu)$ and $\int_X (G + g_{n_{k_l}}) \to \int_X G$. By the first extension of DCT, $\{f_{n_{k_l}}\}$ converge in L^1 to f. Therefore, for every subsequence of $\{\int_X |f_n - f|\}$, there is a further subsequence converging to 0, hence by Urysohn's subsequence principle, f_n converge in L^1 to f. □

Theorem 10. *(Fast L^1 convergence) Suppose f_n, f are measurable real/complex valued function on a measured space (X, μ) such that $\sum_{n\geq 1} \|f_n - f\|_1 < \infty$, then f_n converge pointwise a.e. and almost uniformly to f.*

Proof. Set $E_{N,m} = \{x | \exists n \geq N : |f_n(x) - f(x)| \geq 1/m\}$, then $E_{N,m} \subseteq \cup_{n\geq N}\{x : |f_n(x) - f(x)| \geq 1/m\}$, hence $\mu(E_{N,m}) \leq \sum_{n\geq N} m \|f_n - f\|_1 < \infty$. By fast convergence, this measure can be made arbitrarily small. Now we repeat the argument in the proof of Egorov's theorem.

Given $\epsilon > 0$, for every m, choose N_m such that for $n \geq N_m, \mu(E_{n,m}) \leq \epsilon/2^m$ and take $E = \cup_{m\geq 1} E_{N_m,m}$. Then, $\mu(E) \leq \epsilon$ and if $x \notin E$, then it is not in every $E_{N_m,m}$, hence for $n \geq N_m, |f_n(x) - f(x)| \leq 1/m$ and this N_m doesn't depend on x, therefore f_n converges uniformly to f outside A. □

Theorem 11. *On a measured space (X, μ), let f_n, f be measurable real/complex valued functions. Then f_n converge in measure to f if and only if every subsequence f_{n_k} has a further subsequence converging almost uniformly to f.*

Proof. First suppose f_n converge in measure to f. In this case, it suffices to show that there is a subsequence converging almost uniformly to f because every subsequence also converges in measure to f. For each m, obtain n_m such that

$$\forall n \geq n_m, \mu(\{x : |f_n(x) - f(x)| > 1/m\}) \leq 1/2^m.$$

We may take $n_1 < n_2 < \dots$. Set $E_m = \{x : |f_{n_m}(x) - f(x)| > 1/m\}$, then $\mu(E_m) < 1/2^m$. We claim that f_{n_k} converge almost uniformly to f.

Now, given $\epsilon > 0$, choose M large enough such that $\sum_{n \geq M} \mu(E_n) < \epsilon$, such an M exists because the sum $\sum 2^{-n}$ converges. Set $E = \cup_{n \geq M} E_n$. Given any $\eta > 0$, choose $N \geq M$ such that $1/N < \eta$. If $x \notin E$, then $x \notin E_k$ for $k > N$, hence $|f_{n_k}(x) - f(x)| < 1/k < \eta$, therefore f_{n_k} converge uniformly outside E, and E has measure at most ϵ. Thus a subsequence converges almost uniformly.

Conversely suppose every subsequence f_{n_k} has a further subsequence converging almost uniformly to f. If f_n doesn't converge to f in measure, then there is some $\epsilon_0 > 0$ such that $\mu(\{x : |f_n(x) - f(x)| > \epsilon_0\}) \not\to 0$, so there is some $\epsilon_1 > 0$ such that for infinitely many $n, \mu(\{x : |f_n(x) - f(x)| > \epsilon_0\}) > \epsilon_1$. Obtain a subsequence $\{f_{n_k}\}$ satisfying this condition, then there is a further subsequence $\{f_{n_{k_l}}\}$ converging almost uniformly to f.

Then, we can obtain a set E of measure at most $\epsilon_1/2$ such that $f_{n_{k_l}}$ converges uniformly outside E, hence there is some L such that for $l \geq L, |f_{n_{k_l}}(x) - f(x)| < \epsilon_0/2 \, \forall x \notin E$. However, by the construction of the initial subsequence,

$$\{x : |f_{n_{k_l}}(x) - f(x)| > \epsilon_0\} \subseteq E \text{ and } \mu(\{x : |f_{n_{k_l}}(x) - f(x)| > \epsilon_0\}) > \epsilon_1$$

which is a contradiction. Therefore, f_n converge in measure to f. □

The domination condition plays an important role in going from one mode of convergence to another as seen above. There is a more general condition called uniform integrability. For simplicity, suppose we have a sequence of functions that we want to say converge to zero. The problem comes when the convergence happens because the f_n escape to infinity in one of the ways mentioned earlier. We can control this escape by the condition of uniform integrability.

Definition 1. *Let (X, μ) be a measured space. A sequence $\{f_n\}$ of absolutely integrable measurable functions is said to be uniformly integrable if*

- $\sup \|f_n\|_1 < \infty$.
- $\sup_n \int_{|f_n| > M} |f| \to 0$ *as* $M \to \infty$. *This condition prevents f from escaping vertically.*
- $\sup_n \int_{|f_n| < \delta} |f| \to 0$ *as* $\delta \to 0$. *This condition prevents f from spreading horizontally.*

Lemma 4. *If a sequence of absolutely integrable function f_n is dominated by a $g \in L^1$, then $\{f_n\}$ is uniformly integrable.*

Proof. We have $|f_n| \leq g$, so $\sup \|f_n\|_1 \leq \|g\|_1 < \infty$. Next, given M, we have

$$\int_{|f_n| > M} |f_n| \leq \int_{|f_n| > M} g \leq \int_{g > M} g$$

and the sequence $g\chi_{g \leq M}$ increases to g pointwise, so by dominated convergence theorem, $\int_{g \leq M} g \to \int_X g$ as $M \to \infty$, hence $\int_{g > M} g \to 0$ as $M \to \infty$.

Lastly,

$$\int_{|f_n| < \delta} |f_n| \leq \int_{|f_n| < \delta} \min(g, \delta) \leq \int_X \min(g, \delta)$$

and the sequence $\min(g, \delta)$ converges to 0 pointwise, so by dominated convergence theorem the right side goes to 0 as $\delta \to 0$. Therefore, $\{f_n\}$ is uniformly integrable. □

For some other sufficient conditions for uniform integrability, see [1].

Lemma 5. *Suppose f_n are uniformly integrable on a measured space (X,μ), then for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\int_E |f_n| d\mu \leq \epsilon$$

whenever $n \geq 1$ and E is a measurable set with $\mu(E) \leq \delta$.

Proof. let E be any measurable set, then

$$\begin{aligned}\int_E |f_n| d\mu &= \int_{E\cap|f_n|>M} |f_n| d\mu + \int_{E\cap|f_n|\leq M} |f_n| d\mu \\ &\leq \int_{|f_n|>M} d\mu + M\mu(E)\end{aligned}$$

Given $\epsilon > 0$, by uniform integrability, we may choose M large enough such that the first term is $< \epsilon/2$. We can suitably choose a $\delta > 0$ such that $M\delta < \epsilon/2$, then for any measurable E we have $\int_E |f_n| \leq \epsilon$ as required. □

Theorem 12. *(Uniform integrable convergence in measure) On a measured space X, let f_n be a sequence of uniformly integrable functions and f another measurable function. Then f_n converge in L^1 to f if and only if they converge in measure.*

Proof. If it converges in L^1, then we know that it converges in measure by Chebyshev-Markov inequality. Assume it converges in measure. By uniform integrability, there is an $A > 0$ such that $\|f_n\|_1 \leq A \forall n \geq 1$. We know from a previous theorem that there is a subsequence of f_n converging pointwise a.e. to f, hence by Fatou's lemma, we conclude that $\|f\|_1 \leq A$, therefore f is absolutely integrable.

We need to show that $\|f_n - f\|_1$ converges to 0. To show that, we will divide the integral $\int_X |f_n - f|$ into two parts as

$$\int_X |f - f_n| d\mu = \int_{|f_n - f|\geq\kappa} |f_n - f| d\mu + \int_{|f_n-f|<\kappa} |f_n - f| d\mu$$

for some suitable $\kappa > 0$ such that both terms on the right can be made small.

Next, given $\epsilon > 0$, by uniform integrability, there is a $\delta > 0$ such that

$$\int_{|f_n|<\delta} |f_n| d\mu \leq \epsilon \,\forall\, n.$$

Since f is absolutely integrable, applying the dominated convergence theorem to $f\chi_{|f|<\delta}$ and shrinking δ if necessary, we also have

$$\int_{|f|<\delta} |f| d\mu \leq \epsilon.$$

Let $0 < \kappa < \delta/2$ be a small quantity which we will choose appropriately later. We then have

$$\int_{|f_n-f|<\kappa;|f|\leq\delta/2} |f_n| d\mu \leq \epsilon \text{ and } \int_{|f_n-f|<\kappa;|f|\leq\delta/2} |f| d\mu \leq \epsilon$$

hence by the triangle inequality,

$$\int_{|f_n-f|<\kappa;|f|\leq\delta/2} |f - f_n| d\mu \leq 2\epsilon.$$

Lastly, by Chebyshev-Markov inequality,

$$\int_{|f_n-f|<\kappa;|f|>\delta/2} |f_n - f| d\mu \leq \frac{A}{\delta/2}\kappa.$$

Shrinking κ if necessary, we may make the right side $\leq \epsilon$. Combining everything above, we get

$$\int_{|f_n - f| < \kappa} |f_n - f| \leq 3\epsilon.$$

Next we tackle the region where $|f_n - f| \geq \kappa$. Here we use the fact that f_n converge in measure to f, so there is an $N \geq 1$ such that for $n \geq N, \mu(\{x : |f_n(x) - f(x)| \geq \kappa\}) \leq \kappa$. Using the lemma above and shrinking κ if necessary, we have for $n \geq N$

$$\int_{f_n - f| \geq \kappa} |f_n| d\mu \leq \epsilon \text{ and } \int_{f_n - f| \geq \kappa} |f| d\mu \leq \epsilon$$

hence by triangle inequality,

$$\int_{|f_n - f| \geq \kappa} |f_n - f| \leq 2\epsilon.$$

Therefore, for $n \geq N, \|f_n - f\|_1 \leq 5\epsilon$ and hence f_n converge in L^1 to f. $\square$

We have considered five main modes of convergence, pointwise a.e. (AE), almost uniformly (AU), uniformly (U), in measure (M) and in L^p norm. Under various hypothesis, we have various relations between these modes of convergences. We summarize them below.

Under no additional hypothesis:

- U $\implies$ AU $\implies$ AE
- U $\implies L^\infty \implies$ AU and U outside a null set
- AU $\implies$ M
- $L^1 \implies$ M (Chebyshev-Markov)

Under finite measure:

- AE $\implies$ AU (Egorov's theorem)
- $L^{p_1} \implies L^{p_2}$ when $1 \leq p_2 \leq p_1 \leq \infty$.

Different domination conditions; subsequences:

- f_n, g_n converge a.e. to f, g respectively and $|f_n| \leq g_n$ a.e. and $\int g_n \to \int g$ with $g_n, g \in L^1$, then $f_n \to f$ in L^1
- M $\implies$ subsequence converging AE
- M+dominated by absolutely integrable $\implies L^1$
- AE+fast convergence $\implies$ AU
- M $\iff$ every sybsequence has a further subsequence converging AU
- Under uniform integrability, $L^1 \iff$ M

There are many more modes of convergence. For example, we haven't discussed convergence in distribution which is defined in the context of probability spaces or the relation various L^p modes of convergence have with other modes. However, what we have discussed is quite general and interesting enough to merit this discussion.

References

[1] An Introduction to Measure Theory, Terence Tao

[2] Measure and integral(PDF), E. Kowalski